PMM U.S.S.R.,Vol.51,No.5,pp.593-597,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE STABILITY OF STATIONARY MOTIONS OF BODIES WITH SPHERICAL INERTIA TENSOR IN A NEWTONIAN FORCE FIELD*

R.S. SULIKASHVILI

The problem of the motion of an unconstrained rigid body with a spherical inertia tensor in a Newtonian force field is considered. Stationary motions are determined for the homogeneous bodies of simplest form (a cube, cone, and cylinder) and their stability is investigated. The motions determined and their stability are in full agreement with the analogous results obtained in /1/**(**In the third paragraph of the precis of /1/ the word "apex" should be replaced by "base".) for the stationary motions of bodies clamped at the centre of mass.

1. Let us introduce a fixed system of coordinates $O\xi_0\eta_0\zeta_0$ with origin at the centre of attraction O, and a moving system of coordinates $Gx_1x_2x_3$ with origin at the centre of mass G of the body and its axes directed along its principal central axes of inertia. Let us also introduce the system of coordinates $O\xi\eta\zeta$ obtained from $O\xi_0\eta_0\zeta_0$ by rotating the latter about the axis $O\eta_0 = O\eta$ by an angle σ. We denote the unit vectors of the ξ, η, ζ axes by α, β, γ; $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$ are their projections on the x_i axes.

The position of the body in the fixed system of coordinate axes will be characterized by the angle σ, the coordinates ξ, η, ζ of its centre of mass, and by the direction cosines $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$. Let m_1 be the mass of the centre of attraction, m the mass of the body, f the gravitational constant, $\mu = fm_1$, A, B, C the principal central moments of inertia of the body, and ω_i the projections on the x_i axes of the instantaneous velocity vector of the body in its motion relative to the $O\xi\eta\zeta$ system. The variables σ, ξ, η, ζ are redundant, and we shall therefore assume that $\xi = 0$, i.e. that the centre of mass of the body lies in the $O\eta\zeta$ plane which rotates with an angular velocity of $\sigma^{\cdot} = d\sigma/dt$ about the $O\eta$ axis.

The kinetic energy T of the body is given by the expression

$$2T = m(\eta^{\cdot 2} + \zeta^{\cdot 2} + \zeta^2\sigma^{\cdot 2}) + \sigma^{\cdot 2}(A\beta_1^2 + B\beta_2^2 + C\beta_3^2) +$$
$$2\sigma^{\cdot}(A\omega_1\beta_1 + B\omega_2\beta_2 + C\omega_2\beta_3) + (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$
$$\omega_1 = \psi^{\cdot}\sin\theta\sin\varphi + \theta^{\cdot}\cos\varphi, \quad \omega_2 = \psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi$$
$$\omega_2 = \psi^{\cdot}\cos\theta + \varphi^{\cdot}$$

where φ, ψ, θ are Euler angles determining the orientation of the body relative to the $O\xi\eta\zeta$ axes.

The equations of motion of the body can be written in the form of Lagrange equations, using the variables $\eta, \zeta, \sigma, \varphi, \psi, \theta$ as the generalized coordinates q_i. The equations admit of the energy integral $T + \Pi = \text{const} = h$, where Π is the potential energy of Newtonian attraction, and a cyclic integral expressing the constancy of the projection of the angular momentum of the body on the $O\eta$ axis. The latter integral yields the following expression for the cyclic velocity:

$$S\sigma^{\cdot} = K - 2(A\omega_1\beta_1 + B\omega_2\beta_2 + C\omega_3\beta_3), \quad K = \text{const} \tag{1.1}$$
$$S = m\zeta^2 + (A\beta_1^2 + B\beta_2^2 + C\beta_2^2)$$

where S is the moment of inertia of the body about the $O\eta$ axis.

Ignoring the cyclic coordinate σ, we construct the Routh function

$$R_0 = L - \sigma^{\circ}K = R_2 + R_1 - W_0, \quad W_0 = K^2/(2S) + \Pi$$

Using Routh's theorem we can reduce the problem of determining the stationary motions of the body and the analysis of their stability to the study of the stationary values of the modified potential energy W_0.

Let $\tau = K/(2h)$ be the characteristic time, $R = [K^2/(2m_1h)]^{1/2}$ the characteristic size of the orbit, $l = [(A + B + C)/(3m)]^{1/2}$ the characteristic dimension of the bodies and $l/R = \varepsilon$ a small parameter. We introduce the dimensionless variables and parameters as follows:

*Prikl.Matem.Mekhan.,51,5,758-762,1987

$$M = mM', \quad t = \tau t', \quad \xi = R\xi', \quad \eta = R\eta', \quad \zeta = R\zeta'$$
$$x_i = lx_i' = \varepsilon R x_i' \quad (i = 1, 2, 3)$$

and henceforth omit the primes.

We have the following expression (in dimensionless form) for the function Π :

$$\Pi = -\mu \int\limits_{(M)} \frac{dm}{\Delta} = -\frac{\mu}{R} \int\limits_{(M)} F(\varepsilon)\, dm$$
$$\Delta = R\,[(\xi + \varepsilon b_\alpha)^2 + (\eta + \varepsilon b_\beta)^2 + (\zeta + \varepsilon b_\gamma)^2]^{1/2}$$
$$b_\delta = x_1\delta_1 + x_2\delta_2 + x_3\delta_3, \quad \delta = \alpha, \beta, \gamma; \quad F(\varepsilon) = R/\Delta$$

Expanding Π in powers of ε, we obtain

$$\Pi = -\frac{\mu}{R}\left(I_0 + \varepsilon I_1 + \frac{\varepsilon^2}{2} I_2 + \frac{\varepsilon^3}{6} I_3 + \frac{\varepsilon^4}{24} I_4 + \cdots\right) \tag{1.2}$$
$$I_j = \int\limits_{(M)} F^j(0)\, dm \quad (j = 0, 1, 2, 3, 4)$$

Below we shall give the values of the coefficients in (1.2) for a cube, cone and cylinder.

2. We have, for a body in the form of a cube (the x_i axes are parallel to the cube faces) with edge a (for the dimensionless form $a = \sqrt{6}$)

$$I_0 = \frac{M}{r}, \quad I_1 = I_2 = I_3 = 0, \quad I_4 = 7\,\frac{5r^2 - 2\varkappa}{16r^9} - \frac{133}{80r^5}$$

Then we obtain the following expression for the function W_0 :

$$W_0 = -E\left[\frac{1}{r} + \frac{k}{r^5}\varepsilon^4\right] + \Pi_\sigma$$
$$\Pi_\sigma = K^2/(2S), \quad r = (\eta^2 + \zeta^2)^{1/2}, \quad Q = \mu M/R$$
$$k = (21 - 35\varkappa)/960, \quad \varkappa = u_1^4 + u_2^4 + u_3^4$$
$$u_i = (\eta\beta_i + \zeta\gamma_i)/r \quad (i = 1, 2, 3)$$
$$\psi_1 = u_1^2 + u_2^2 + u_3^2 - 1 = 0$$

The problem of determining the stationary motions of the body reduces to that of determining an unconditional extremum of the function $W = W_0 + \lambda\psi_1$ where λ is the undetermined Lagrange multiplier.

The equations of stationary motions have the form

$$\frac{\partial W}{\partial \eta} = E\eta\left(\frac{1}{r^3} + \frac{5k}{r^7}\varepsilon^4\right) = 0 \tag{2.1}$$
$$\frac{\partial W}{\partial \zeta} = E\zeta\left(\frac{1}{r^3} + \frac{5k}{r^7}\varepsilon^4\right) - \frac{MK^2\zeta}{S^2} = 0$$
$$\frac{\partial W}{\partial u_i} = 2\lambda_0 u_i^3 + 2\lambda u_i = 0 \quad (i = 1, 2, 3), \quad \lambda_0 = \frac{7\varepsilon^4}{96r^5}$$

and admit of the following families of solutions:

$$\eta = 0, \quad \zeta = \zeta_0 = N(1 + D), \quad u_1 = 0, \quad u_2 = \pm 1/\sqrt{2}, \quad u_3 = \pm 1/\sqrt{2} \quad (123) \tag{2.2}$$
$$\eta = 0, \quad \zeta = \zeta_0 = N(1 + 16D), \quad u = u_2 = 0, \quad u_3 = \pm 1 \quad (123) \tag{2.3}$$
$$\eta = 0, \quad \zeta = \zeta_0 = N(1 + 6D), \quad u_1 = \pm 1/\sqrt{3}, \quad u_2 = \pm 1/\sqrt{3}, \quad u_3 = \pm 1/\sqrt{3} \tag{2.4}$$
$$N = RK^2/(\mu M^2), \quad D = 72A^2\varepsilon^4/(7M^2)$$

Here λ has the following corresponding values for (2.2)-(2.4): $-{}^1/_2\lambda_0 E$; $-\lambda_0 E$; $-{}^1/_3\lambda_0 E$.

Eqs.(2.1) have no other solutions.

The solutions (2.2)-(2.4) correspond to the relative equilibria of the body in a circular orbit of radius ζ_0, whose centre of mass moves with constant orbital angular velocity $\omega_0 = K_0/S_0$ (see (1.1)) given by the condition

$$\omega_0\zeta_0^2 = \mu R^{-1}(1 - k\varepsilon^4/\zeta_0^4)$$

(the zero subscript means that the corresponding quantity is calculated for the solutions (2.2)-(2.4)).

For the solutions (2.2) the radius vector $OG = \zeta_0$ of the centre of attraction is parallel to a diagonal of one of the faces of the cube, for (2.3) it is parallel to one of the sides, and for (2.4) it is directed along one of the diagonals of the cube.

Let us investigate the stability of the motions (2.2)-(2.4). We introduce the notation

$$a_{11}=\left(\frac{\partial^2 M}{\partial\eta^2}\right)_0, \quad a_{22}=\left(\frac{\partial^2 W}{\partial\zeta^2}\right)_0, \quad a_{12}=a_{21}=\left(\frac{\partial^2 W}{\partial\eta\partial\zeta}\right)_0$$
$$a_{33}=\left(\frac{\partial^2 W}{\partial u_1^2}\right)_0, \quad a_{44}=\left(\frac{\partial^2 W}{\partial u_2^2}\right)_0, \quad a_{23}=a_{32}=\left(\frac{\partial^2 W}{\partial\zeta\partial u_1}\right)_0$$
$$a_{55}=\left(\frac{\partial^2 W}{\partial u_3^2}\right)_0, \quad a_{24}=a_{42}=\left(\frac{\partial^2 W}{\partial\zeta\partial u_2}\right)_0, \quad a_{25}=a_{52}=\left(\frac{\partial^2 W}{\partial\zeta\partial u_3}\right)_0$$

The remaining partial second-order derivatives are equal to zero.

Let us denote by $(\delta^2 W)$ the value of the quadratic form $\delta^2 W$ on the linear manifold $\delta\psi_1 = 0$. The eigenvalues of the quadratic form $(\delta^2 W)$ can be called the stability coefficients, and the number of negative roots the degree of instability (we shall denote it by χ).

The values of a_{22} are the same for all solutions (2.2)-(2.4)

$$a_{22} = M\omega_0^2 (4M\zeta_0 - S_0)/S_0 - 2E/\zeta_0^3 + O(\varepsilon^4)$$

and the values of the other a_{ij} differ.

For the solutions (2.2) we have

$$a_{11}=E\,\frac{384\zeta_0^4-7\varepsilon^4}{384\zeta_0^7}, \quad a_{33}=-\frac{7E\varepsilon^4}{96\sqrt{2}\,\zeta_0^5}$$
$$a_{44} = a_{55} = -2\sqrt{2}a_{33}, \quad a_{24} = a_{25} = 5a_{33}/\zeta_0$$

(the remaining partial second-order derivatives are equal to zero). The conditions of positive definiteness of $(\delta^2 W)$ are expressed by the inequalities

$$a_{55} > 0, \quad a_{33}a_{44} > 0, \quad a_{11}a_{33}a_{44} > 0, \quad a_{11}a_{33}a_{24}(a_{44} - a_{24}) > 0$$

the second and third of which do not hold. Therefore for the solutions (2.2) we have $\chi = 2$.

For (2.3) we have

$$a_{11} = E\,(96\zeta_0^4 - 7\varepsilon^4)/(96\zeta_0^7), \quad a_{44} = -7E\varepsilon^4/(48\zeta_0^5)$$

and the sufficient conditions are expressed by the inequalities

$$a_{44} > 0, \quad a_{44}^2 > 0, \quad a_{11}a_{44}^2 > 0, \quad a_{11}a_{22}a_{44}^2 > 0$$

the first of which does not hold. Therefore for these motions we have $\chi = 2$.

This in the case of the motions (2.2) and (2.3) Routh's theorem does not lead to a definite conclusion concerning the stability.

For the solutions (2.4) we have

$$a_{11} = E\,(144\zeta_0^4 + 7\varepsilon^4)/(144\zeta_0^7), \quad a_{33} = 35E\varepsilon^4/(288\zeta_0^5)$$

and the conditions of stability reduce to the inequalities

$$a_{11} > 0, \quad a_{22} > 0, \quad a_{33} > 0 \tag{2.5}$$

which are always satisfied and we have for them $\chi = 0$. The motions (2.4) are stable with respect to the variables $\eta, \zeta, u_i, \eta^{\cdot}, \zeta^{\cdot}, \sigma^{\cdot}, \omega_i$ $(i = 1, 2, 3)$.

3. Calculating the terms of the expansion (1.1) for the bodies in the form of a cone and cylinder, of radii a and heights $2a$ and $\sqrt{3}a$ respectively (the height of the bodies is determined from the conditions $A = B = C$, and in dimensionless form $a = 1$), we obtain the following expressions for the modified potential energy of these bodies:

$$W=-E\left[\frac{1}{r}+\frac{u_3(3-5u_3^2)}{16r^4}\varepsilon^3\right]+\Pi_\sigma \text{ for a cone}$$
$$W=-E\left[\frac{1}{r}+11\,\frac{5u_3^2(6-7u_3^2)-3}{640r^5}\varepsilon^4\right]+\Pi_\sigma \text{ for a cylinder}$$

The equations of stationary motions of the cone and cylinder $\delta W = 0$ yield the following families of solutions:

$$\eta = 0, \quad \zeta = \zeta_0 = N, \quad u_3 = \mp 1 \tag{3.1}$$
$$\eta = 0, \quad \zeta = \zeta_0 = N, \quad u_3 = \mp 1/\sqrt{5} \tag{3.2}$$

for the cone, and

$$\eta=0,\quad \zeta=\zeta_0=N\left(1-\frac{128}{465}D_1\right),\quad u_3=\mp 1 \tag{3.3}$$

$$\eta=0,\quad \zeta=\zeta_0=N\left(1-\frac{64}{335}D_1\right),\quad u_3=0 \tag{3.4}$$

$$\eta=0,\quad \zeta=\zeta_0=N\left(1-\frac{384}{8885}D_1\right),\quad u_3=\mp\frac{\sqrt{3}}{\sqrt{7}} \tag{3.5}$$

$$D_1 = A^2\varepsilon^4/M^2$$

for the cylinder.

The solutions (3.1)-(3.5) correspond to the stationary motions of the bodies whose centres of mass move along a circular orbit and the bodies themselves are in equilbrium with respect to the $O\xi\eta\zeta$ system of coordinates. For the solutions (3.1), (3.3) the vector OG is directed along the axis of symmetry of the bodies, and for (3.4) it is perpendicular to the axis of symmetry. We note that in the case of (3.1) the cone points towards the centre of attraction with its apex when $u_3 = -1$, and with its base when $u_3 = 1$. The solutions (3.2), (3.5) are such that the vector OG passes across the boundary circumference of the base.

Let us investigate the stability of the motions (3.1)-(3.5). We have the following expressions for the second-order partial derivatives of the function W for the cone, for the values (3.1), (3.2):

$$a_{11}=\left(\frac{\partial^2 W}{\partial\eta^2}\right)_0=E\,\frac{2\zeta_0^3+\varepsilon^3}{2\zeta_0^6},\quad a_{33}=\left(\frac{\partial^2 W}{\partial u_3^2}\right)_0=\pm\frac{3E\varepsilon^3}{4}$$

for the solutions (3.1) and

$$a_{11}=E\,\frac{2\sqrt{5}\,\zeta_0^3\mp\varepsilon^3}{2\sqrt{5}\,\zeta_0^6},\quad a_{33}=\mp\frac{3E\varepsilon^3}{2\sqrt{5}}$$

for the solutions (3.2), and we have $a_{22} = (\partial^2W/\partial\zeta^2)_0 = E/\zeta_0^3$ for all solutions. The remaining second-order partial derivatives are equal to zero.

The conditions of stability for solutions (3.1), (3.2) reduce to the inequalities (2.5), the first two of which hold and the last of which does not hold for solution (3.1) when $u_3 = 1$, and for (3.2) when $u_3 = -1/\sqrt{5}$. This leads us to conclusion that the motions are unstable $(\chi = 1)$, while the remaining motions (3.1), (3.2) are stable $(\chi = 0)$ with respect to the variables

$$\eta,\ \zeta,\ u_3,\ \eta^\cdot,\ \zeta^\cdot,\ \sigma^\cdot,\ \omega_1,\ \omega_2,\ \omega_3 \tag{3.6}$$

We note that the stability and instability of the body in the form of a cone changes when it is rotated by $180°$ about the normal to the plane of the orbit. The result has no analogue for the usual satellite approximation.

The second-order partial derivatives of the function W for a cylinder, for the values (3.3)-(3.5), are:

$$a_{11}=\left(\frac{\partial^2 W}{\partial\eta^2}\right)_0=E\,\frac{80\zeta_0^4-11\varepsilon^4}{80\zeta_0^7},\quad a_{33}=\left(\frac{\partial^2 W}{\partial u_3^2}\right)_0=-\frac{11E\varepsilon^4}{2\zeta_0^5}$$

for solutions (3.3),

$$a_{11}=E\,\frac{640\zeta_0^4-33\varepsilon^4}{640\zeta_0^7},\quad a_{33}=-\frac{33E\varepsilon^4}{8\zeta_0^5}$$

for solutions (3.4) and

$$a_{ss}=E\,\frac{560\zeta_0^4+33\varepsilon^4}{560\zeta_0^7},\quad a_{33}=\frac{627E\varepsilon^4}{196\zeta_0^5}$$

for solutions (3.5), and $a_{22} = (\partial^2W/\partial\zeta)_0 = a_{11} + O(\varepsilon^4)$ for all solutions, with the remaining second-order partial derivatives equal to zero.

The conditions of stability for solutions (3.3)-(3.5) reduce to the inequalities (2.5), the first two of which hold, and the last of which does not hold for solutions (3.3) and (3.4). Therefore, we can conclude that these motions are unstable $(\chi = 1)$, and the motions (3.5) are stable $(\chi = 0)$ with respect to the variables (3.6).

It should be noted that even a small deviation from the linear dimensions of the bodies (e.g. the height of the cone) can lead to the appearance of new dynamic effects not included in the present formulation of the problem.

REFERENCES

1. SULIKASHVILI R.S., On the effect of third- and fourth-order moments of inertia on the motion of a rigid body. PMM 51, 2, 1987.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.5,pp.597-600,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

EQUATIONS OF MOTION OF A CARRIER SUPPORTING DYNAMICALLY UNBALANCED AND ASYMMETRIC FLYWHEELS IN AN INERTIAL MEDIUM*

V.A. KONOPLEV

The methods described in /1-5/ are used to derive the equations of motion of a body supporting dynamically unbalanced and asymmetric flywheels in an inertial fluid. The equations combine the accuracy of inclusion of inertial effects with the compactness of matrix notation, with the convenience of constructing the computational procedures based on modern matrix processing facilities of the digital computer without resorting to the scalar equations. The equations obtained are used to formulate a problem of programmed rotation of flywheels, ensuring that the carrier moves as required, provided it exists.

The equations obtained can be used for a straightforward investigation of the motion of a vibrating table under the condition that the position, the inertial characteristics and the modes of motion are all known, and for determining the above characteristics which ensure that the table moves in a prescribed manner (the control problem).

We shall use, for simplicity, a single symbol $E_k = (O_k, [e^k])$ for all rigid bodies of the system, and for the associated Cartesian systems coordinate with the origin O_k and an orthonormed basis, $[e^k] = (e_1^k, e_2^k, e_3^k)$, $e_1^k = \| 1, 0, 0 \|^T$, $e_2^k = \| 0, 1, 0 \|^T$, $e_3^k = \| 0, 0, 1 \|^T$, so that E_0 will denote the inertial coordinate system, E_1 is the body of the carrier, E_p $(p = 2, 3, \ldots, n)$ the instruments under test installed on E_1, E_s $(s = 2, 3, \ldots, m)$ are the flywheels, including those which may be mounted on the instruments under test.

The dynamic screw of such a system is described in E_0 in the form

$$Z_0^0 = \mathbf{L}_1^{00} Z_1^1, \quad \mathbf{L}_1^{00}: E_0 \to E_1 \tag{1}$$

$$Z_1^1 = \mathbf{K}_1^1 \dot{V}_1^{01} + \sum_{s=1}^{m} \mathbf{L}_s^{11} \boldsymbol{\Theta}_s^s f_s \varphi_s^{\cdot} \tag{2}$$

$$\mathbf{K}_1^1 = \boldsymbol{\Theta}_1^1 + \sum_{p=2}^{n} \mathbf{L}_p^{11} \boldsymbol{\Theta}_p^p \mathbf{L}_p^{11,\,T} + \sum_{s=2}^{m} \mathbf{L}_s^{11} \boldsymbol{\Theta}_s^s \mathbf{L}_s^{11,\,T} + \boldsymbol{\Lambda}_{11}^{11} + \boldsymbol{\Lambda}_{p1}^{p1}$$

Here Z_1^1 is the same screw in E_1; $\mathbf{L}_1^{00} = \mathbf{T}_1^{00} [C_1^0]$ is a (6×6)-matrix situated in E_0 /1/, and

$$\mathbf{T}_1^{00} = \begin{Vmatrix} E & 0 \\ \langle O_1^0 \rangle^0 & E \end{Vmatrix}, \quad [C_1^0] = \begin{Vmatrix} C_1^0 & 0 \\ 0 & C_1^0 \end{Vmatrix} \tag{3}$$

where $\langle O_1^0 \rangle^0$ is a skew symmetric (3×3) matrix generated by the position vector O_1^{00} of O_1 in E_0 on the basis $[e^0]$; E is a unit (3×3) matrix, $C_1^0 = C_3(\psi_1) C_2(\theta_1) C_1(\varphi_1)$ is a (3×3) matrix of the orientation $[e^1]$ on $[e^0]$, $([e^1] = [e^0] C_1^0)$ is the simplest (3×3) matrix of

*Prikl.Matem.Mekhan.,51,5,763-766,1987